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Time Reversal of Quantum Experiments

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Time reversal exchanges the roles of the initial and final stages of an experiment. This fact is properly represented here by an alternative time-reversal transformation in quantum theory. In elementary quantum experiments one prepares a system, lets it propagate over time, and checks for a particular value of a complete sequence of system variables. Following the operational interpretation of quantum theory, the initial and final stages of such experiments are represented by kets and bras. Hence, the new time-reversal transformation maps kets into bras and vice versa. Wigner's result about changes of description of a quantum system is extended so as to include transformations between kets and bras. Invariance of the Schwinger action principle under time reversal requires the new time-reversal transformation to be linear. In this paper the time reversal of experiments is represented completely, whereas Wigner's formulation only applies to the propagation phase (so-called time evolution) of an experiment.

1. INTRODUCTION

When describing time reversal, Wigner (1959, §26) assumed the interpretation of quantum mechanics called orthodox by him and employed by most textbooks today. I call this interpretation the "state interpretation of quantum mechanics" because it clings to classical thought in that it takes for granted that a system always is in some state of being (Finkelstein, 1995, §5.6.3). As is done in classical physics, Wigner associated each state with a so-called time-reversed state. Given a (continuous) dynamical sequence of states in time which satisfies the equation of motion, the reversely ordered sequence of the time-reversed states satisfies the same equation of motion (which in case of T violation is governed by a somewhat different Hamiltonian.)

In the context of the state interpretation of quantum mechanics, Wigner's time reversal suffices to describe the time reversal of the so-called undisturbed time evolution of a quantum system. However, any quantum experiment

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starts with a preparation of a quantum system, continues with a propagation process (so-called undisturbed time evolution) of the system, and ends with a measurement on it. Wigner's formulation does not suffice to represent the time reversal of experiments completely, in that his notion of time reversal cannot be applied to the quantum system's spontaneous transition from the propagation process to the measurement outtake of the experiment. In this paper I formulate the time reversal of complete experiments. This formulation is based on the operational interpretation of quantum theory, according to which kets and bras represent different kinds of acts of the experimenter.

I restrict myself to experiments whose propagation phase is not interrupted by filtering or other kinds of acts of the experimenter. Among these experiments I mainly concern myself with 'elementary' ones: In an elementary experiment the preparation of the system by the experimenter is maximal (Finkelstein, 1995, §§1.2.1, 1.3.2, 2.1), i.e., describable by a value of some complete sequence of system variables; and the measurement is a maximal test, i.e., a test for a particular value of some complete sequence of system variables. (A more complex measurement, such as of the position of a particle, can be regarded as the combination of parallel conducted tests, such as for particular position values.) The operations of the experimenter that start and end an experiment have, respectively, been called "preparation of the system in a state" and "test" by Giles (1970, Section 3), "preparation" and "registration" by Ludwig (1983, 1985), "pre- and postselection" by Aharonov and Vaidman (1991), "initial" and "final acts" or "input" and "outtake operations" by Finkelstein (1995, §1.2.1). In this paper I reserve the terms input and outtake operations for maximally specified ones. Such input and outtake operations are represented by elements of dual vector spaces. These elements are called ket and bra vectors, respectively. In contrast, according to the state interpretation of quantum mechanics employed by Wigner and most textbooks today, kets and bras indiscriminately represent system "states" in the sense of states of being rather than modes of doing.

Concretely, an attempted outtage operation, i.e., a test, consists in presenting a quantum system with a filter and second with a detector. Let us say that the outtake operation occurs, or that the test is positive, if the quantum system is detected. Let us call the experiment successful if the outtake operation occurs.

My formulation of time reversal is based on the insight that the time reversal of an occurring outtake operation (represented by a bra) is an input operation (represented by a ket) and vice versa (Section 2). I extend Wigner's result about changes of description of a quantum system so as to include transformations between kets and bras (Section 3). We then find that time reversal is either represented by a unitary or by an antiunitary map T from kets to bras (and by its adjoint $T^{\dagger}$). The question whether T is linear or antilinear is decided by requiring that the Schwinger action principle be invariant under time reversal (Section 4). In order to fix T completely, we need to supply the transformation behavior of a complete sequence of variables of the system under study. The transformation property of the Hamiltonian and the canonical momenta will be derived from their generator properties (which are implied by the Schwinger action principle).

A point $\mathfrak{t}$ of the time axis $\mathbb{T}$ often will be denoted by its coordinate representation with respect to a particular time variable t:

$$\mathfrak{t} = t : t' \tag{1.1}$$

The statement that the variable t takes on the value t' is abbreviated into $t : t'$.

Equations that only hold if the picture-independent Schwinger action principle is satisfied are called weak equations and distinguished from more general equations by using the symbol $\doteq$ rather than $=$.

2. COMPLETE DESCRIPTIONS OF AN EXPERIMENT AND OF ITS TIME REVERSAL

The time-reversal transformation depends on the chosen time axis. I restrict myself to experiments that are conducted by an inertial experimenter. Hence, we only need to deal with a single time axis, so that the framework of quantum mechanics suffices. I employ a picture-independent formulation of quantum mechanics. Input operations at different times then are represented by elements of ket vector spaces that are fibers of a bundle over the time axis. Dually, outtake operations are represented by elements of the dual bra bundle. The propagation process between the input and outtake operations is represented by a connection U (Mantke, 1995, Section 2; Asorey *et al.*, 1982).

A connection U transports a ket $|\psi\rangle \in I(\mathfrak{t}_1)$ at time $\mathfrak{t}_1$ into kets $U(\mathfrak{t}_2, \mathfrak{t}_1)|\psi\rangle \in I(\mathfrak{t}_2)$ at other times $\mathfrak{t}_2$, dually a bra $\langle\varphi| \in I^D(\mathfrak{t}_1)$ into $\langle\varphi|U(\mathfrak{t}_1, \mathfrak{t}_2) \in I^D(\mathfrak{t}_2)$, and has the composition property

$$U(\mathfrak{t}_3, \mathfrak{t}_1) = U(\mathfrak{t}_3, \mathfrak{t}_2)U(\mathfrak{t}_2, \mathfrak{t}_1) \tag{2.1}$$

The restriction of a connection U to a time interval $[\mathfrak{t}_1, \mathfrak{t}_2]$ will be denoted by $U_{\mathfrak{t}_1}^{\mathfrak{t}_2}$,

$$U_{\mathfrak{t}_1}^{\mathfrak{t}_2} = \{U(\mathfrak{t}'', \mathfrak{t}') |\, \mathfrak{t}'', \mathfrak{t}' \in [\mathfrak{t}_1, \mathfrak{t}_2]\} \tag{2.2}$$

The connection $U_{\mathfrak{t}_1}^{\mathfrak{t}_2}$ over a time interval $[\mathfrak{t}_1, \mathfrak{t}_2]$ describes a possible propagation process from $\mathfrak{t}_1$ to $\mathfrak{t}_2$ maximally: the most detailed information about the propagation process that can be obtained experimentally is its effect on transition probabilities for experiments that start and end within the time interval $[\mathfrak{t}_1, \mathfrak{t}_2]$. Evidently, all this information is contained in the connection over this time interval.

I assume a positive-definite metric of the ket spaces. The associated adjoint operator † maps kets into bras, and vice versa, and is antiunitary,

$$\langle\psi| \in I^D(t) \overset{\dagger}{\leftrightarrow} |\psi\rangle \in (t) \tag{2.3}$$

$$\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^* \overset{\dagger}{\leftrightarrow} \langle\varphi|\psi\rangle \tag{2.4}$$

$|\psi\rangle$ and $\langle\psi|$, as well as the operations they represent, we call adjoint to one another. The transition of a quantum system from an input phase of an experiment to the adjoint outtake phase is compulsory.

An elementary experiment $\mathscr{E}$ starts with an input operation, continues with a propagation process, and ends with an outtake operation, so that it is represented by a triplet

$$E = (\langle\varphi|, U_{t_1}^{t_2}, |\psi\rangle), \qquad \langle\varphi| \in I^D(t_2), \quad |\psi\rangle \in I(t_1) \tag{2.5}$$

The time reversal of an experiment $\mathscr{V}$ is a hypothetical experiment $\mathscr{V}_T$ that would closely imitate a reversed film of the original experiment. If an elementary experiment $\mathscr{E}$ is successful, the input and outtake operations both occur. A film of an elementary experiment $\mathscr{E}$ run backward then makes the outtake operation of the experimenter look like an input operation, and vice versa. Hence, the time reversal $\mathscr{E}_T$ of the successful experiment $\mathscr{E}$ is represented by the following kind of triplet:

$$E_T = (\langle\psi_T|, (U_T)_{t_2}^{t_1}, |\varphi_{T^D}\rangle), \qquad \langle\psi_T| \in I^D(t_1), \quad |\varphi_{T^D}\rangle \in I(t_2) \tag{2.6}$$

The preparations of the successful experiments $\mathscr{E}$ and $\mathscr{E}_T$, respectively, need to be regarded as time reversals of one another. E and E_T also represent these preparations of the experiments, regardless of whether $\mathscr{E}$ and $\mathscr{E}_T$ are successful. Hence, the experiments $\mathscr{E}$ and $\mathscr{E}_T$, as represented by E and E_T, should be regarded as time reversals of one another, regardless of whether the experiments are successful.

To be more general, let us consider a successful experiment $\mathscr{V}$ that starts with a maximal input operation, continues with a propagation process, and ends with a non-maximally specified outtake operation, so that it is represented by a triplet

$$V = (P, U_{t_1}^{t_2}, |\psi\rangle), \qquad P^2 = P = P^\dagger \in I(t_2) \otimes I^D(t_2), \quad |\psi\rangle \in I(t_1) \tag{2.7}$$

In any orthogonal basis of the subspace $I^D(t_2)P \subset I^D(t_2)$ the projector can be expanded as

$$P = \sum_{n=1}^{d} |\varphi_n\rangle\langle\varphi_n| \tag{2.8}$$

The experiment $\mathcal{V}$ can be regarded as a parallel array of the elementary experiments $\mathcal{E}_n$,

$$E_n = (\langle\varphi_n|\ U_{t_1}^{t_2}, |\psi\rangle), \qquad \langle\varphi_n| \in I^D(t_2), \quad |\psi\rangle \in I(t_1), \quad n = 1, \ldots, d \tag{2.9}$$

provided they share the same input operation, propagation process, and outtake detector. The dimension d of the orthogonal complement measures the multitude of possibilities the quantum system has to exit the experimental region and to reach the detector. This multitude can be implemented in a time-reversed experiment by inputting d systems into d experimental regions. Accordingly, the time-reversed experiment $\mathcal{V}_T$ is composed of the time-reversed elementary experiments $\mathcal{E}_{nT}$,

$$E_{nT} = (\langle\psi_T|,\ (U_T)_{t_2}^{t_1}, |\varphi_{nT^D}\rangle),$$
$$\langle\psi_T| \in I^D(t_1),\ |\varphi_{nT^D}\rangle \in I(t_2), \quad n = 1, \ldots, d \tag{2.10}$$

In the experiment $\mathcal{V}_T$, d systems are prepared parallely, in manners represented by the various kets $|\varphi_{nT^D}\rangle$. The parallel experiments $\mathcal{E}_{nT}$ shall share the same outtake detector, which shall be sensitive to whether any system left an experimental region through the outtake channel represented by $\langle\psi_T|$, but not to how many systems have done so.

We have seen now how the problem of time reversal of general experiments can be reduced to the time reversal of elementary ones. With equation (2.6) we obtained the general form of the transformation law of elementary experiments; in particular, the ketvector at t_1 is mapped into a bravector at t_1, and the bravector at t_2 is mapped into a ketvector at t_2.

For a complete determination of the time-reversal transformation, one requires a concept of time-reversal invariance of a system dynamics. We call the dynamics of the system under study time-reversal invariant if any elementary experiment $\mathcal{E}$ and the associated experiment $\mathcal{E}_{\text{test}\_T}$, to be defined in the following, have the same transition probability. To define $\mathcal{E}_{\text{test}\_T}$ one first replaces the input operation of $\mathcal{E}$ with the time-reversed outtake operation of $\mathcal{E}$, and dually the outtake operation with the time-reversed input operation. The new input and outtake operations are represented by $|\varphi_T\rangle \in I(t_2)$ and $\langle\psi_T| \in I(t_1)^D$, respectively. In order to construct $\mathcal{E}_{\text{test}\_T}$ out of these operations, we require a connection to parallel transport them to the reversed times. Then the outtake operation, as in the original experiment, succeeds the input operation after the time span $\Delta t' = t(t_2) - t(t_1)$. A connection is equivalent to a picture (Mantke, 1995, Section 2). I propose that the connection appropriate for the definition of $\mathscr{E}_{\text{test}-T}$ is equivalent to the 'standard' picture $B(\chi)$. In the case of a mechanical system, let q denote the sequence of mechanical configuration operators. Then $B(\chi)$, also called the configuration picture, is defined by the following property: In $B(\chi)$ the eigenkets (and dually eigenbras) of the configuration operators q at different times with equal eigenvalues are represented by the same vector at the chosen reference time t_0,

$$\forall \mathrm{t} \in \mathrm{T} \quad |q : q', \mathrm{t}\rangle^{\chi} = |q : q', \mathrm{t}_0\rangle \in I(\mathrm{t}_0) \tag{2.11}$$

[In case q are Cartesian coordinates, $B(\chi)$ is the Schrödinger picture.] In the case of a field system, $q(\sigma)$ denotes the uncountable sequence of field operators on a spacelike surface. Like Schwinger (1951, Section II), let us employ complete sequences $\zeta(\sigma)$ of commuting operators, similarly constructed from $q(\sigma)$ for each spacelike surface σ. I restrict myself to those spacelike surfaces σ_{t} that consist of simultaneous points relative to the chosen time axis. In the standard picture $B(\chi)$ of the field system, an equation like (2.11) holds for the eigenvectors of the complete sequences $\zeta(\sigma_{\mathrm{t}})$.

The propagation process of $\mathscr{E}_{\text{test}-T}$ shall be governed by the original dynamics, which is represented by the connection U. Let U^{χ} denote the representation of U in the configuration picture. In this picture $\mathscr{E}_{\text{test}-T}$ is represented by

$$E^{\chi}_{\text{test}-T} = ({}^{\chi}\langle\psi_T|, (U^{\chi})^{-\mathrm{t}_1}_{-\mathrm{t}_2}, |\varphi_T\rangle^{\chi}), \qquad {}^{\chi}\langle\psi_T| \in I^D(\mathrm{t}_0), \quad |\varphi_{T^D}\rangle^{\chi} \in I(\mathrm{t}_0)$$

where

$$t(-\mathrm{t}) = -t(\mathrm{t}) \tag{2.12}$$

If the transition probability of $\mathscr{E}_{\text{test}-T}$ is equal to the one of $\mathscr{E}$,

$$\frac{|\langle\varphi|U(\mathrm{t}_2, \mathrm{t}_1)|\psi\rangle|^2}{\langle\varphi|\varphi\rangle\langle\psi|\psi\rangle} = \frac{|{}^{\chi}\langle\psi_T|U^{\chi}(-\mathrm{t}_1, -\mathrm{t}_2)|\varphi_{T^D}\rangle^{\chi}|^2}{{}^{\chi}\langle\psi_T|\psi_T\rangle^{\chi}\,{}^{\chi}\langle\varphi_{T^D}|\varphi_{T^D}\rangle^{\chi}} \tag{2.13}$$

we say that the dynamics is time-reversal invariant for the quantum process that occurs in the successful experiment $\mathscr{E}$. If this holds for all elementary experiments $\mathscr{E}$, the dynamics is completely time-reversal invariant.

We have seen that the concept of time-reversal invariance and hence the time-reversal transformation are linked to a particular picture.

The time reversal linked to the configuration frame will be determined by requiring that it leave the Schwinger action principle invariant.

3. AN EXTENSION OF WIGNER'S RESULT ABOUT REPRESENTATION CHANGES

In order to prepare the invariance considerations of the action principle, I extend Wigner's result about representation changes (Wigner, 1931, §XX.4; 1959, §§20.4, 26).

In this discussion I suppress the time index τ. An input and an outtake operation are respectively represented by a ray of ketvectors and a ray of bravectors, or by nonzero elements of these rays. In the preceding section it became clear that the modes, input and outtake operation, are not necessarily invariant under representation changes of experiments: Under representation changes that involve time reversal, input operations become outtake operations and vice versa; other representation changes do not change the input or outtake mode. In any case, the representation changes of input and outtake operations are transformations of ketrays in I and brarays in I^D into rays in dual vector spaces, respectively, denoted as I' and I'^D. Here I' is a Hilbert space isomorphic to I. When time reversal is involved, $I' = I^D$, otherwise $I' = I$.

A ray of ketvectors can be represented by one of its nonzero elements. For any nonzero $|\psi\rangle \in I$ which represents some ketray let us choose an element $|\psi_W\rangle \in I'$ of the transformed ketray that is normalized like $|\psi\rangle$,

$$\langle\psi|\psi\rangle = \langle\psi_W|\psi_W\rangle \tag{3.1}$$

Next we define a not necessarily linear operator W on I to be the function that maps a nonzero $|\psi\rangle \in I$ into the associated $|\psi_W\rangle \in I'$ and the zero ket into itself,

$$0, |\psi\rangle \neq 0 \in I, \qquad W|\psi\rangle = |\psi_W\rangle \in I', \qquad W0 = 0' \in I' \tag{3.2}$$

(W stands for Wechsel, the German word for change. Wigner used the symbol O_R instead, because his argument was sparked by the question of representation change under a rotation.) The transition from an input operation to its adjoint outtake operation is compulsory. This also holds after a representation change, including time reversal. Hence, (2.3) should transform into

$$\langle\psi_{W^D}| \in I'^D \overset{\dagger}{\leftrightarrow} |\psi_W\rangle \in I' \tag{3.3}$$

Let $\langle\varphi|$ and $\langle\varphi_{W^D}|$ denote bras that represent the same outtake operation before and after the representation change, respectively. Because of (3.3) we can drop the superscript D of W in $\langle\varphi_{W^D}|$. This, together with (3.2), gives[2]

[2] $W^\dagger$ as defined here has the following property: For $\langle\varphi| \in I^D$ and $|\alpha\rangle \in I'$

$$\langle\varphi|W^\dagger|\alpha\rangle = \langle\varphi_W|\alpha\rangle = \langle\alpha|\varphi_W\rangle^* = \langle\alpha|W|\varphi\rangle^*$$

(Later, W will be either a linear or an antilinear operator.) If W is an antilinear operator, my definition of the adjoint of W needs to be distinguished from a definition, employed, for example, by Nachtmann (1986, §4.5.3), according to which

$$\langle\varphi|W^\dagger(\text{Nachtmann})|\alpha\rangle = \langle\alpha|W|\varphi\rangle$$

Moreover, if W is antiunitary, $W^\dagger$ as defined by me is not equal to W^{-1}, in contrast to

$$W^\dagger(\text{Nachtmann}) = W^{-1}$$

$$\langle \varphi | W^\dagger = \langle \varphi_W | = \langle \varphi_{W^D} | \in I'^D \tag{3.4}$$

Like Wigner, let us assume that the probability for instantaneous transitions is invariant under representation changes. In terms of our associated ray representors this assumption is equivalent to

$$|\langle \varphi | \psi \rangle| = |\langle \varphi_W | \psi_W \rangle| = |\langle \varphi | W^\dagger W | \psi \rangle| \tag{3.5}$$

Furthermore, Wigner proposed that W should map an orthogonal basis into an orthogonal basis. He then showed that for each $|\psi\rangle$ the phase of the associated $|\psi_W\rangle$ can be chosen so that W becomes either a unitary or an antiunitary (Wigner, 1959, §§20.4, 26) operator. Wigner assumed that W is a map from I to I. Nevertheless, his argument is also valid if W is a map from I to $I' \cong I$.

Hence, I can use his result for my time-reversal transformation, which maps kets into bras and vice versa: One can fix the phases of the transformed ket and bra vectors in (2.6) such that they are obtained by operators T and $T^\dagger$ that are either unitary or antiunitary,

$$\langle \psi_T | = T | \psi \rangle \in I^D(\mathrm{t}), \qquad | \varphi_{T^D} \rangle = \langle \varphi | T^\dagger = | \varphi_T \rangle \in I(\mathrm{t})$$
$$T \text{ unitary or antiunitary} \tag{3.6}$$

In order to discuss the representation change of experiments that include a propagation process, I need to reintroduce time indices. W, e.g., T, is a smooth family of operators $W(\mathrm{t})$ parametrized by time,

$$|\psi\rangle \in I(\mathrm{t}), \qquad |\psi_W\rangle = W|\psi\rangle = W(\mathrm{t})|\psi\rangle \in I'(\mathrm{t}) \tag{3.7}$$

Next, I derive the connection U_W which represents the propagation processes after a representation change. As for instantaneous transitions, I assume that the transition probability for experiments with a propagation phase is invariant under representation changes,

$$|\langle \varphi | U(\mathrm{t}_2, \mathrm{t}_1) | \psi \rangle| = |\langle \varphi_W | U_W(\mathrm{t}_2, \mathrm{t}_1) | \psi_W \rangle| = |\langle \varphi_W | U_W(\mathrm{t}_2, \mathrm{t}_1) W | \psi \rangle| \tag{3.8}$$

According to (3.5), the left side is also equal to

$$|\langle \varphi | U(\mathrm{t}_2, \mathrm{t}_1) | \psi \rangle| = |\langle \varphi_W | W U(\mathrm{t}_2, \mathrm{t}_1) | \psi \rangle| \tag{3.9}$$

Because (3.8) and (3.9) hold for all $|\psi\rangle \in I(\mathrm{t})$ and for all $\langle \varphi_W | \in I'^D(\mathrm{t})$,

$$e^{i\phi(\mathrm{t}_2,\mathrm{t}_1)} U_W(\mathrm{t}_2, \mathrm{t}_1) W = W U(\mathrm{t}_2, \mathrm{t}_1) \tag{3.10}$$

This equation needs to hold for all times t_1 and t_2. The transformed connection U_W also needs to satisfy the composition law (2.1), as does U. Hence, the phase factor needs to have the composition property, too, so that we can absorb it into the transformed connection,

$$U_W(t_2, t_1) = WU(t_2, t_1)W^{-1} \tag{3.11}$$

The inverse of W exists, because W is either unitary or antiunitary. Furthermore, U_W is a unitary connection, because the product of three unitary operators, or of one unitary and two antiunitary ones, is unitary (Wigner, 1959, §26). Thus, $U_W(t_2, t_1) = U_W(t_2, t_1)^{\dagger -1}$ and we can bring equation (3.11) into the alternative form

$$U_W(t_2, t_1) = W^{\dagger -1}U(t_2, t_1)W^{\dagger} \tag{3.12}$$

(Note that if W is antiunitary, $W^{\dagger} \neq W^{-1}$. See footnote 2.) The transition amplitude is invariant or transformed into its complex conjugate, depending on whether W is unitary or antiunitary,

$$\begin{aligned} \langle \varphi_W | U_W(t_2, t_1) | \psi_W \rangle &= \langle \varphi | U(t_2, t_1) | \psi \rangle, && W \text{ unitary} \\ &= \langle \varphi | U(t_2, t_1) | \psi \rangle^*, && W \text{ antiunitary} \end{aligned} \tag{3.13}$$

To apply equation (3.11) to time reversal, we need to know the right and left actions of $U_W(t_2, t_1)$ in terms of mappings. To be more general, let us transform a map A from $I(t')$ to $I(t'')$ into the map $W(A)$ from $I'(t')$ to $I'(t'')$,

$$|\alpha\rangle \in I'(t'), \qquad W(A)|\alpha\rangle = WAW^{-1}|\alpha\rangle \equiv W \circ A \circ W^{-1}(|\alpha\rangle) \in I'(t'') \tag{3.14}$$

where $\circ$ denotes the composition of mappings. The left action of A, the so-called pullback A^*, is a map from $I^D(t'')$ to $I^D(t')$,

$$\langle \varphi | \in I^D(t''), \qquad \langle \varphi | A \equiv A^*(\langle \varphi |) \in I^D(t') \tag{3.15}$$

Similarly, the left action or pullback of $W(t)$ is a map $W^*(t)$ from $I'^D(t)$ to $I^D(t)$. The left action or pullback of $W(A)$ is

$$\begin{aligned} \langle \beta | \in I'^D(t''), \qquad \langle \beta | W(A) &\equiv W(A)^*(\langle \beta |) \\ &= W^{*-1} \\ &\circ A^* \\ &\circ W^*(\langle \beta |) \in I'^D(t') \end{aligned} \tag{3.16}$$

4. TIME-REVERSAL INVARIANCE OF THE PICTURE-INDEPENDENT SCHWINGER ACTION PRINCIPLE

According to (3.6), (3.12), (3.14), and (3.16), the representations of elementary experiments (2.5) transform either linearly or antilinearly into

$$E_T = (\langle \psi_T |, (U_T)_{t_2}^{t_1}, | \varphi_T \rangle)$$

where

$$\langle\psi_T| = T|\psi\rangle, \qquad |\varphi_T\rangle = \langle\varphi|T^\dagger$$
$$(U_T)_{t_2}^{t_1} = \{U_T(t', t'') = T(U(t'', t'))\,|\, t'', t' \in [t_1, t_2]\} \tag{4.1}$$

For the definition of $T(U(t'', t'))$, recall that I use the ket notation for the images of ket vectors under a general representation transformation W, (3.2), whereas I use the bra notation for the ones of T, (3.6). Consider a general map A from $I(t')$ to $I(t'')$. In accordance with (3.14), $T(A)$ therefore is the following left action on bra vectors:

$$\langle\alpha| \in I^D(t'), \qquad \langle\alpha|T(A) \equiv T(A)(\langle\alpha|) = T \circ A \circ T^{-1}(\langle\alpha|) \in I^D(t'') \tag{4.2a}$$

In accordance with (3.16), the right action or pullback of $T(A)$ is

$$|\beta\rangle \in I(t''), \qquad T(A)|\beta\rangle \equiv T(A)^*(|\beta\rangle) = T^{*-1} \circ A^* \circ T^*(|\beta\rangle) \in I(t') \tag{4.2b}$$

In light of (3.8) and (4.1), the *a priori* condition (2.13) for the system dynamics to be invariant under time reversal can be brought into the form

$$U_T^\chi(t_1, t_2) = U^\chi(-t_2, -t_1)e^{i\theta(t_1,t_2)} = U^{-1\chi}(-t_1, -t_2)e^{i\theta(t_1,t_2)} \tag{4.3}$$

The index χ indicates that this equation holds only in the configuration picture. The phase factor has the composition property, and hence θ can be written as

$$\theta(t_1, t_2) = \varphi(t_1) - \varphi(t_2), \qquad \varphi(t) := \theta(t, t_r) \tag{4.4}$$

where t_r is an arbitrary reference time. The invariance condition now becomes

$$U_T^\chi(t_1, t_2) = e^{-i\varphi(t_2)}U^\chi(-t_2, -t_1)e^{i\varphi(t_1)} \tag{4.5}$$

We are now in the position to ensure the invariance of the picture-independent Schwinger action principle (Mantke, 1992, 1995, Section 5).[3] This principle is an extension of the action principle by Schwinger (1960, 1970),[4] who formulated his principle in the Heisenberg picture. The picture-independent quantum action is a functional that maps a connection $U_{t_1}^{t_2}$ over a time interval $[t_1, t_2]$ into $I(t_2) \otimes I^D(t_1)$,

[3] In Mantke (1992) I failed to notice that the picture-independent Schwinger action does not act linearly on the dynamical connection because of the square velocity term of the Lagrangian.
[4] Schwinger (1960) includes time variations into the class of dynamical changes, whereas I call variations of time kinematical, as I do variations of configuration.

$$W[P_{t_1}^{t_2}] := \int_{t(t_1)}^{t(t_2)} dt'\, P(t_2, t : t')L(t', q, \dot{q}_{t,P})P(t : t', t_1) \tag{4.6}$$

q is a smooth family of linear operators $q(t)$ on the fibers $I(t)$. Its connection derivative is the family of derivatives

$$\dot{q}_{t,P}(t : t') = \frac{d}{dt''}\bigg|_{t''=t'} P(t : t', t : t'')q(t : t'')P(t : t'', t : t') \tag{4.7}$$

The picture-independent form of the Schwinger action principle for a mechanical system is

$$\begin{aligned} &\delta\langle q : q'', t_2 | U(t_2, t_1) | q : q', t_1\rangle \\ &\doteq \frac{i}{\hbar}\langle q : q'', t_2 | \delta W[U_{t_1}^{t_2}] | q : q', t_1\rangle \end{aligned} \tag{4.8a}$$

[In the case of a field system one uses the eigenvectors of the complete sequences $\zeta(\sigma_t)$ introduced below equation (2.11) (Schwinger, 1951, Section II).] A few words are in order here to describe the variations involved in this equation. Say we express the functional W in terms of the variables $t^* = t + \delta t(t)$ (including dt^*) and $q^* = q + \delta q(t)$. The resulting form variation of W is equal to δW. Similarly, the variation of the transition amplitude results from replacing the eigenket $|q : q', t : t'\rangle$ of q at time $t_1 = t : t'$ by the eigenket

$$|q^* : q', t^* : t'\rangle = |q : (q' - \delta q(t)), t : (t' - \delta t(t))\rangle \tag{4.8b}$$

of q^* at time $t^* : t' = t : (t' - \delta t(t))$ with equal eigenvalue, and from replacing the eigenbra in the dual fashion. This completes the definition of the allowed kinematical variations. The dynamical variation of the transition amplitude is the matrix element of the dynamical variation of the evolution operator. The dynamical variation of the action is the form variation which represents the change of dynamics. For a given action, the action principle implies a particular phase convention for the configuration eigenvectors [Mantke (1992), equations (V.3.19), (V.3.28), and (V.6.18)]. Adding a total time derivative of a function of t and q to the Lagrangian changes this phase convention, but not the dynamics (Mantke, 1992, Section V.4).

I restrict the discussion for a while to a system with a time-reversal-invariant dynamics. According to the action principle, the functional form of the action determines the system dynamics. The invariance of the dynamics therefore requires that the transformed action is of the same form as the original action, up to a total time derivative of a function of t and q in the action integral. I choose the phases of the eigenvectors in the transformed action principle such that this total time derivative is zero,

$$W_T[(P_T)^{t_1}_{t_2}] = \int_{t_T(t_2)}^{t_T(t_1)} dt_T' \; P_T(t_1, t : t_T')L_T(t_T', q_T, [\dot{q}_T]_{t_T,P_T})P_T(t : t_T', t_2)$$

$$= \int_{t_T(t_2)}^{t_T(t_1)} dt_T' \; P_T(t_1, t : t_T')L(t_T', q_T, [\dot{q}_T]_{t_T,P_T})P_T(t : t_T', t_2) \quad (4.9)$$

Hence,

$$L_T(\cdot, \cdot, \cdot) = L(\cdot, \cdot, \cdot) \quad (4.10)$$

$t_T = -t$, and q_T is defined by

$$Tq|\psi\rangle = \langle\psi_T|q_T \quad (4.11)$$

Consequently, our time-reversed variables are

$$t_T = -t, \qquad q_T = T(q) \quad (4.12)$$

For the definition of $T(\cdot)$ see (4.2).

According to equations (4.1) and (4.7), the transformed connection derivative is

$$[\dot{q}_T]_{t_T,P_T} = -[\dot{q}_T]_{t,P_T} = -T(\dot{q}_{t,P}) \quad (4.13)$$

It is well known that a system with the Lagrangian

$$L(t, q, v) = \tfrac{1}{2}vmv - V(q) \quad (4.14)$$

is time-reversal invariant, i.e., passes the test described at the end of Section 2. With the help of (4.9), (4.12), and (4.13) we obtain for the time-reversed Lagrangian

$$L_T(t_T', q_T, [\dot{q}_T]_{t_T,P_T}) = L(t_T', q_T, [\dot{q}_T]_{t_T,P_T})$$

$$= \tfrac{1}{2}[\dot{q}_T]_{t_T,P_T} m [\dot{q}_T]_{t_T,P_T} - V(q_T)$$

Therefore,

$$L_T(t_T', q_T, [\dot{q}_T]_{t_T,P_T}) = T(L(t', q, \dot{q}_{t,P})) \quad (4.15)$$

The action of a connection over a time interval then transforms tensorially, too,

$$W_T[(P_T)^{t_1}_{t_2}] = T(W[P^{t_2}_{t_1}]) \quad (4.16)$$

This, together with (4.1), shows that the picture-independent Schwinger action principle (4.8a) transforms into

$$\delta\langle[q : q'', t_1]_T|U_T(t_1, t_2)|[q : q', t_2]_T\rangle$$

$$\doteq \frac{i}{\hbar}\langle[q : q'', t_1]_T|\,\delta W_T[(U_T)^{t_1}_{t_2}]\,|[q : q', t_2]_T\rangle, \qquad T \text{ unitary} \quad (4.17)$$

$$\delta\langle [q : q'', \mathfrak{t}_1]_T | U_T(\mathfrak{t}_1, \mathfrak{t}_2) | [q : q', \mathfrak{t}_2]_T \rangle^*$$
$$\doteq \frac{i}{\hbar} \langle [q : q'', \mathfrak{t}_1]_T | \, \delta W_T[(U_T)^{\mathfrak{t}_1}_{\mathfrak{t}_2}] \, | [q : q', \mathfrak{t}_2]_T \rangle^* \qquad T \text{ antiunitary}$$

We conclude that for the action principle to be invariant, T needs to be a unitary map from bras to kets. The unitarity of T means that its adjoint is equal to the inverse of its pullback,

$$T^\dagger = T^{*-1} \tag{4.18}$$

Let us now return to general, possibly noninvariant, actions. Equation (4.16) needs to hold also for noninvariant actions, so that the action principle remains invariant under T, i.e., so that (4.17) holds. A general action hence transforms as

$$T\colon \quad W \mapsto W_T$$

with

$$W_T[P^{\mathfrak{t}_2}_{\mathfrak{t}_1}] = \int_{t_T(\mathfrak{t}_1)}^{t_T(\mathfrak{t}_2)} dt'_T \, P(\mathfrak{t}_2, t : t'_T) L_T(t'_T, q_T, [\dot{q}_T]_{t_T,P}) P(t : t'_T, \mathfrak{t}_1) \tag{4.19a}$$

where the transformed Lagrangian is defined by equation (4.15). The Lagrangian is an expression of coordinates and their connection derivatives. Equations (4.12), (4.13), and (4.15) thus imply

$$L_T(t', q, v) = L(-t', q, -v) \tag{4.19b}$$

which we expected from classical mechanics.

So far my discussion has been quite general. q may have been a sequence of mechanical configuration operators, as in equations (2.11) and (4.14), or an uncountable sequence of field operators on a spacelike surface. In order to fix the time-reversal transformation completely, we need to know how q transforms. I perform the remaining analysis for a mechanical system, the analogous analysis for a field analysis being similar.

The configuration operators of a mechanical system are invariant,

$$q_T = q \tag{4.20}$$

The configuration eigenkets transform into configuration eigenbras and vice versa. In order to fix the phases of the transformed configuration eigenvectors, I—for the second time in this section—refer to an invariant action W_{inv}. The phase convention associated with an action [see below equation (4.8b)] is invariant whenever the action is invariant. This phase convention fixes the phase of the configuration eigenvectors up to a global phase. Let us rephase the

T operator such that this global phase is invariant as well. The configuration eigenvectors, whose phases are adapted to W_{inv}, then are invariant,

$$T|\text{t}, q : q'\rangle_{W_{\text{inv}}} = {}_{W_{\text{inv}}}\langle \text{t}, q : q'|, \qquad {}_{W_{\text{inv}}}\langle \text{t}, q : q'|T^{\dagger} = |\text{t}, q : q'\rangle_{W_{\text{inv}}} \tag{4.21}$$

and I have succeeded in determining T. In general, the phase conventions associated respectively with W_{inv} and a time-reversal noninvariant action W differ from one another, so that equations (4.21) do not hold for the convention associated with W.

I continue with a derivation of the transformation law of the generators p and H, and close with the consequences of time-reversal invariance of an action.

Note that, by considering kinematic variations, the picture-independent Schwinger action principle implies (Mantke, 1992, Section V.6),

$$\begin{aligned} &|t : (t' + dt'), q : (q' + dq')\rangle \\ &\doteq U(t : t' + dt', t : t')\left(1 + \frac{i}{\hbar} H\, dt' - \frac{i}{\hbar} p\, dq'\right)|t : t', q : q'\rangle \end{aligned} \tag{4.22a}$$

where

$$p := \frac{\partial L}{\partial v}, \qquad H(t', q, p) \equiv H(t : t') := D_v(\dot{q}_{t,U})\cdot L - L \tag{4.22b}$$

$D_x(u)$, called the polarization operation, generates the linear change of a function f of operators $a, \ldots x, \ldots$ due to a variation u of the argument x (Finkelstein, 1955),

$$D_x(u)\cdot f(a, \ldots, x, \ldots) := \lim_{\epsilon\to 0} \frac{f(a, \ldots, x + \epsilon u, \ldots) - f(a, \ldots, x, \ldots)}{\epsilon}$$

For the configuration eigenbras, the action principle implies the adjoint of equation (4.22a). Equation (4.22a) implies that in the configuration picture the dynamical connection is generated by the Hamiltonian

$$U^{\chi}(t : t' + dt', t : t') \doteq 1 - \frac{i}{\hbar} H^{\chi}(t : t')\, dt' \tag{4.23}$$

The time-reversed action principle, equation (4.17), implies the time-reversed versions of equation (4.22a) and its adjoint in terms of variables p_T and H_T defined analogously to equation (4.22b). Applying T to equation (4.22a) and substituting $-dt'_T$ for dt', one finds the transformation law of the generators p and H,

$$p_T \doteq -T(p), \qquad H_T \doteq T(H) \tag{4.24}$$

For the transformed Hamiltonian function we obtain

$$H_T(t'_T, q_T, p_T) = H(-t'_T, q_T, -p_T) \tag{4.25}$$

If the dynamics of the system is invariant under time reversal, the transformed Lagrangian function is equal to the original one,

$$L(t', q, v) = L_T(t', q, v) = L(-t', q, -v) \tag{4.10'}$$

Then the Hamiltonian function of equation (4.22b) is invariant as well,

$$H(t', q, p) = H_T(t', q, p) = H(-t', q, -p) \tag{4.26}$$

Since the Hamiltonian operator generates the dynamical connection in the configuration picture, the invariance of the Hamiltonian function implies the invariance of the configuration representation of the connection,

$$U^{\times}(t : t'', t : t') \doteq U^{\times}(t_T : t'', t_T : t') \tag{4.27}$$

Note that this symmetry is in accordance with the *a priori* invariance conditions (4.5) and (2.13).

Moreover, for an invariant action the phase convention of the configuration eigenvectors [see Eq. (4.21)] and hence the generator p in Eq. (4.22a) do not change under time reversal,

$$p \doteq p_T \doteq -T(p) \tag{4.28}$$

Unless the Hamiltonian contains odd powers of $t'p$, the invariance condition (4.25) implies that H is an even function of t' as well as of p, so that the Hamiltonian operator is invariant,

$$H \doteq H_T \doteq T(H) \tag{4.29}$$

5. CONCLUSIONS

Wigner's antiunitary time-reversal operator T_W is equal to the composition of my unitary operator T, which maps kets into bras, and the adjoint operator †,

$$T_W = \dagger \circ T = T^{\dagger} \circ \dagger \tag{5.1}$$

In terms of Wigner's time-reversal operator, the tensorial transformation, equation (4.2a), of a map A from $I(\mathfrak{t}')$ to $I(\mathfrak{t}'')$ is

$$\langle\alpha| \in I^D(\mathfrak{t}') \qquad \langle\alpha|T(A) \equiv T(A)(\langle\alpha|) = (T_W \circ A \circ T_{W^{-1}})^{\dagger}(\langle\alpha|) \in I^D(\mathfrak{t}'') \tag{5.2a}$$

According to equations (4.18) and (4.2b), the right action or pullback of $T(A)$ is

$$|\beta\rangle \in I(t'') \qquad T(A)|\beta\rangle \equiv T(A)^*(|\beta\rangle) = T^\dagger \circ A^* \circ T^{\dagger^{-1}}(|\beta\rangle)$$
$$= T_W \circ A^{*\dagger} \circ T_{W^{-1}}(|\beta\rangle) \equiv T_W A^\dagger T_{W^{-1}}|\beta\rangle \in I(t') \qquad (5.2b)$$

The latter is familiar from the literature [e.g., Nachtmann (1986), §4.5.3, p. 66, below equation (4-141)].

For determining T, we required an *a priori* concept of time-reversal invariance [equation (2.13)] so that we would be able to identify a time-reversal invariant dynamics and action [equation (4.9)]. The invariance of this action allowed us to determine its transformation law [equation (4.16)] (which we later extended to general, possibly noninvariant, actions). With the help of the transformation law of the invariant action, we were able to show that T is linear rather than antilinear [equation (4.17)].

The dependence of the time-reversal transformation on the choice of a standard frame appeared in the *a priori* definition of time-reversal invariance (end of Section 2), in the transformation law of the variables q [equation (4.20) or field theory analog], and in the deduced invariance condition of the propagation processes [equation (4.27)].

As pointed out, there is a simple relationship between my operator T and Wigner's T_W. Still, the motivation for the derivation of T was quite different from Wigner's. Wigner assumed the state interpretation, called orthodox by him, of quantum mechanics. Accordingly, his time-reversal operator T_W maps a (continuous) dynamical sequence of state vectors in time, which satisfies the equation of motion, into the reversely ordered sequence of the so-called time-reversed state vectors, which satisfies the same equation of motion. That a system shall be in an abstract, invisible "state" is a concept of being, such as "having a position." I am convinced that quantum theory can be better understood if one interprets it in terms of concepts of becoming rather than of being. Hence I based my formulation on the operational interpretation of quantum theory, according to which kets and bras represent modes of performing, namely input and outtake operations of the experimenter. I achieved a complete representation of the time reversal of experiments (Section 2), whereas Wigner's time reversal only applies to the propagation (evolution) phase of an experiment.

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